Plasma Physics Reports, Vol. 27, No. 4, 2001, pp. 278–292. Translated from Fizika Plazmy, Vol. 27, No. 4, 2001, pp. 296–310.
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PLASMA KINETICS

Residual Electron Momentum and Energy in a Gas Ionized by a Short High-Power Laser Pulse

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Received June 1, 2000; in final form, September 19, 2000

Abstract—A study is made of the nonadiabatic dynamics of photoelectrons produced in the interaction of an elliptically polarized, high-power laser pulse with a gas. Expressions for the so-called residual momentum and energy of the electrons (i.e., the mean electron momentum and energy after the passage of the pulse through the gas) are derived. The residual electron momentum and energy are investigated analytically as functions of the gas and laser parameters. A relationship is established between the residual energy and the electron temperature tensor. *© 2001 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

In recent years, the problem of the residual electron energy (REE) (i.e., the problem of what fraction of the energy acquired by an electron at the top of a laser pulse remains in an electron after the passage of a laser pulse) has been extensively discussed in the literature [1, 2]. For a laser pulse propagating through a preionized gas, the REE is negligible. However, for a pulse propagating in a gas and ionizing it, this energy may become substantial because of the nonadiabatic motion of an electron produced during a short-term ionization event in a laser field. As a femtosecond laser pulse propagates in a low-density plasma, the electron heating due to inverse-bremsstrahlung absorption is insignificant; consequently, after the passage of the pulse, the electron energy is mainly determined by the REE.

The study of REE is particularly important for developing X-ray lasers in which a multiply ionized plasma that is strongly nonequilibrium with respect to ionization and recombination serves as an active medium [2]. In such lasers, the degree to which the plasma is nonequilibrium with respect to these processes should be as high as possible; i.e., it is necessary to produce plasmas with the maximum possible ion charge number and minimum possible REE. This problem can be resolved by ionizing a gas with a short (about one hundred femtoseconds) intense ($I_0 > 10^{15}$ W cm^{-2}) laser pulse.

Here, we apply the so-called "two-stage" ionization model. According to this model, the transition of an electron from the bound state to the state of free motion is described in terms of quantum mechanics (by the theory of tunneling ionization) and its subsequent motion in the laser field is described by the classical equations [3, 4]. However, in contrast to [1–3], we assume that free electrons are produced with a nonzero initial momentum $\mathbf{p}_*$ with the probability determined by the corresponding quantum-mechanical distribution (cf. [4]). The momentum and energy of an electron produced in such a manner are governed by its interaction with the laser field and can be deduced from the classical relativistic equations of motion. Then, we can average the resulting electron momentum and energy over the electron ensemble. The relationships between the residual electron momentum (REM), REE, and electron temperature can be derived by comparing the results obtained by one-particle and hydrodynamic approaches.

In the particular case of a linearly polarized laser pulse and under the assumption that the electrons are born with a zero initial momentum, formulas (13) and (12) for the mean energy and longitudinal momentum of an electron (see below) yield formula (13) from [2]. Our purpose here is to generalize the theory of REE developed by Pulsifer *et al.* [2] so as to take into account the distribution of the produced free electrons over their initial momenta and to consider relativistic laser pulses with an arbitrary elliptic polarization. We show that the ensemble-averaged energy and momentum of a free electron in a laser field consist of two parts: first, strongly oscillating components $Q_{\rm van}$ and $\mathbf{P}_{\rm van}$, which vanish after the pulse leaves the plasma, and, second, weakly oscillating components $Q_{\rm fin}$ and $\mathbf{P}_{\rm fin}$, which are just the REE and REM after the passage of the pulse. The formulas derived here for $Q_{\rm fin}$ and $\mathbf{P}_{\rm fin}$ make it possible to clarify the dependence of the REE and REM on the main gas and laser parameters: the ionization potentials, the laser intensity, the degree of elliptic polarization of the laser field, the laser wavelength, and the pulse shape. We show that, when the ionization front duration is as long as several laser field periods, the REE $Q_{\rm fin}$ is substantially higher than the energy $P_{\mathrm{fin}}^2/(2m)$; consequently, Q_{fin} can be regarded as the energy of *disordered* electron motion. We find that the REE is expressed in terms of the sum of the transverse (with respect to the x-axis along which the pulse propagates) pressure tensor elements Π_{yy} and Π_{zz}: $Q_{\mathrm{fin}} = (\Pi_{yy} + \Pi_{zz})/(2n_e) \equiv (T_{yy} + T_{zz})/2$, where n_e is the electron density and T_{yy} and T_{zz} are the electron temperature tensor elements.[1]

2. IONIZATION MODEL

We treat the problem in a one-dimensional approximation; i.e., we consider the electron motion in the vicinity of the laser-pulse axis, assuming that the pulse is wide enough to neglect both transverse electron drift[2] and laser-light diffraction. We also consider a gas with a sufficiently low density such that the nonlinear processes distorting the pulse shape [8–10] occur on time scales much longer than the pulse duration. In this case, the laser field strength along the propagation direction of the pulse depends on the x coordinate only through the combination $x/c–t$, so that the shape of the propagating pulse can be assumed to be unchanged. Consequently, for convenience, we can consider the electron motion near the point $x = 0$, keeping in mind that the results obtained will also pertain to the remaining electrons, because they move in the same field that is only shifted in phase with respect to the point $x = 0$.

At $x = 0$, the rapidly oscillating component $\tilde{\mathbf{E}}(t)$ of the electric field of the laser pulse can be represented as

$$\begin{aligned} \tilde{\mathbf{E}}(t) &= E(t)[\mathbf{e}_y\cos(\omega_0 t) + \eta\mathbf{e}_z\sin(\omega_0 t)], \\ E(t) &= E_0\exp[-(t/\sigma_t)^2], \\ E_0 &= \sqrt{(8\pi/c)I_0/(1+\eta^2)}, \quad \sigma_t = \tau_{FWHM}/\sqrt{2\ln 2}, \end{aligned} \tag{1}$$

where $\mathbf{e}_y$ and $\mathbf{e}_z$ are unit vectors in the y and z directions, $\eta \in [-1, 1]$ is the degree of elliptic polarization ($\eta = 0$ and $|\eta| = 1$ correspond to linear and circular polarizations, respectively), $E(t)$ is the laser field amplitude, E_0 is the maximum laser field amplitude, I_0 is the peak intensity of the pulse, and τ_{FWHM} is the full width at half-maximum (FWHM) of the pulse.

For the above laser pulse parameters, the gas is ionized on a time scale much shorter than the pulse duration. The ionization can be assumed to proceed via the tunneling mechanism when the Keldysh parameter [11] is much smaller than unity, $\gamma = \omega_0\sqrt{2mJ_k}/eE(t^*) \ll 1$,[3] where m and e are the mass of an electron and the absolute value of its charge, ω_0 is the laser frequency, $E(t^*)$ is the electric field amplitude of the laser wave at the time t^* of an ionization event, and J_k is the ionization potential of an ion in the $(k-1)$th ionization state. For the short ($\tau_{FWHM} < 1$ ps) laser pulses under consideration, the electron–ion collision time in a low-density ($n_e < 10^{19}$ cm^{-3}) gas is longer than the pulse duration; consequently, the processes of recombination and impact ionization do not come into play throughout the pulse.

Under the conditions of tunneling ionization, we can assume that the electron shells are ionized successively (starting from the shell farthest from the nucleus). In this case, the number N of electrons originating by the time t per unit volume in the vicinity of the point x under consideration is determined by the equations

$$\begin{aligned} \frac{\partial N}{\partial t} &= S = \sum_{k=1}^{z_n} kS_k = \sum_{k=1}^{z_n} W_k n_{k-1}, \\ \frac{\partial n_k}{\partial t} &= S_k, \quad S_k = -W_{k+1}n_k + W_k n_{k-1}, \\ k &= 1, \ldots, z_n - 1, \quad S_{z_n} = W_{z_n} n_{z_n - 1}, \\ n_0 &= n_{at} - \sum_{k=1}^{z_n} n_k, \end{aligned} \tag{2}$$

where n_k is the density of the ions in the kth ionization state ($k = 0$ corresponds to a neutral atom), which are heavy enough to be regarded as immobile; W_{k+1} is the ionization rate of these ions; $n_{at} = \sum_{k=0}^{z_n} n_k$ is the total ion density (including neutrals); and z_n is the nuclear charge.

Under our conditions, the rate at which an electron produced via tunneling ionization of an ion collides with the potential barrier formed by both the electric field of this ion and the laser field is much higher than the laser frequency. Therefore, we can search for the tunneling ionization rate in the adiabatic approximation [13], i.e., by substituting the absolute value of the instantaneous laser field, $|\tilde{\mathbf{E}}(t)| = E(t)\sqrt{\cos^2(\omega_0 t) + \eta^2\sin^2(\omega_0 t)}$, into the formula for the ionization rate in a constant electric field, which is equal to the tunneling ionization rate in a circularly polarized field. For arbitrary atoms, the latter rate is described by the Ammosov–Delone–Kraĭnov (ADK) formula [14]. Consequently, in the adiabatic

[1] In [5, 6], it was shown that, in this case, free electrons moving in the y and z directions obey Maxwellian velocity distributions with temperatures T_{yy} and T_{zz}.

[2] Simple estimates made in [7] show that the transverse electron drift can be ignored under the condition $\sigma_r/\lambda_0 > 80(\sigma_t/1\ \mathrm{ps})\sqrt{I_0/10^{17}\ \mathrm{W\ cm^{-2}}}$, where λ_0 is the laser wavelength and σ_r and σ_t are the characteristic width and duration of the pulse, respectively.

[3] More precise conditions for tunneling ionization are determined by Ilkov *et al.* [12], who showed that the tunneling mechanism is dominant when $\gamma \le 0.5$.

approximation, the total ionization rate in the field of an arbitrarily polarized laser pulse has the form

$$W_k(t) = \omega_a \frac{e^1 k^2}{2\pi n_*^4}\left(\frac{4e^1 k^3 E_a}{n_*^4 |\tilde{\mathbf{E}}(t)|}\right)^{2n_*-1} \times \exp\left\{-\frac{2}{3}\left(\frac{k}{n_*}\right)^3 \frac{E_{\text{a.u.}}}{|\tilde{\mathbf{E}}(t)|}\right\}, \quad (3)$$

where $n_* = k\sqrt{J_H/J_k}$ is the principal quantum number of an ion in the $(k-1)$th ionization state with the ionization potential J_k, J_H is the ionization potential of a hydrogen atom, $\omega_a \approx 4.1 \times 10^{16}$ s^{-1} is the atomic frequency, $E_a \approx 5.1 \times 10^9$ V cm^{-1} is the atomic electric field, and $e^1 = \exp(1)$. In deriving formula (3), we assumed that the orbital and magnetic quantum numbers are both zero; this assumption is justified, e.g., in [15].

In order to take into account the distribution of the produced electrons over the initial momenta, we need to know not only the total ionization rate W_k but also the differential ionization cross section $\Gamma_k(\mathbf{p}_*)$, i.e., the probability $W_k(t) = \int \Gamma_k(t, \mathbf{p}_*) d^3\mathbf{p}_*$ for an electron with the initial momentum $\mathbf{p}_*$ to originate in a unit momentum interval per unit time. To determine $\Gamma_k(\mathbf{p}_*)$, we turn to the results obtained by Goreslavsky and Popruzhenko [4], who proposed a formula for the distribution of the ionization-produced electrons over their initial velocities. Strictly speaking, this formula applies to a zero-range atomic potential (i.e., to a potential in the form of a δ function). However, the results obtained by Delone and Kraĭnov [15], who derived the Coulomb correction to the ionization probability with allowance for the long-term nature of the atomic potential, show that the coefficient in front of the exponential function in the expression for the differential ionization cross section is independent of the initial momentum of an ionization-produced electron. In view of this fact and taking into account that the distribution derived in [4] correctly reflects the exponential dependence of the differential ionization cross section on the initial electron velocity, we can extend this distribution to the case of a complex atom. In this way, we choose the coefficient in front of the exponential function so as to describe the total ionization rate by the ADK formula (3). As a result, we obtain

$$\Gamma_k(\mathbf{p}_*, t) d^3\mathbf{p} = \frac{1}{\pi}\frac{E_{\text{a.u.}}}{|\tilde{\mathbf{E}}(t)|}\sqrt{\frac{J_k}{J_H}} W_k(|\tilde{\mathbf{E}}(t)|) \times \exp\left[-\frac{p_{*\perp}^2}{2mJ_H}\frac{E_{\text{a.u.}}}{|\tilde{\mathbf{E}}(t)|}\sqrt{\frac{J_k}{J_H}}\right]\delta(p_{*\tilde{\mathrm{E}}}) dp_{*\tilde{\mathrm{E}}}\frac{d^2 p_{*\perp}}{2mJ_H}, \quad (4)$$

where $W_k(|\tilde{\mathbf{E}}(t)|)$ is defined by formula (3) and the δ function reflects the fact that the electron momentum $p_{*\tilde{\mathrm{E}}}$ in the instantaneous direction of the laser field at the time of an ionization event is equal to zero [4]. In the plane perpendicular to this direction, the electrons obey a two-dimensional isotropic distribution over the initial momenta $p_{*\perp}$.

It should be noted that formulas (3) and (4) are valid under the condition $\alpha_k \equiv (J_H/J_k)^{3/2} E/E_{\text{a.u.}} \ll 1$, which may fail to hold for strong laser fields. In sufficiently strong fields, the ionization can exhibit the phenomenon of *stabilization.* In other words, for $\alpha_k > 1$, the stronger the laser field, the lower are both the ionization probability per unit time and the total ionization probability (see, e.g., [15–19]). In [16–19], the stabilization of ionization was calculated for laser pulses with sharp fronts (the rise time of the front being ten atomic times $\tau_a = 1/\omega_a$ or shorter). On the other hand, Kulander *et al.* [18] noted that, for pulses with smoother fronts, the stabilization effect is less pronounced because the rapid ionization of atoms occurs at the pulse front, where $\alpha_k \ll 1$. Our simulations for light gases that are completely ionized by laser pulses with nonrelativistic intensities (except, possibly, for the 1S electron shell) showed that a pulse with a duration longer than ten laser field periods will completely ionize ions in the $(k-1)$th ionization state by the time at which $\alpha_k \gtrsim 10^{-1}$; by this time, the relative concentration of these ions, n_{k-1}/n_{at}, will become lower than 10^{-2} [20]. Consequently, for stronger laser fields, the uncertainty in determining the probability W_k will affect the final results only slightly. Thus, we can conclude that, in our analysis of pulses with rise times longer than several laser field periods, the stabilization effect is insignificant.

Another restriction on formulas (3) and (4) is that they are written in the nonrelativistic limit and are inapplicable to ions with high ionization potentials (i.e., ions that are ionized by relativistic laser fields) [15]. Consequently, in applying our model to relativistic laser pulses, we must assume that the gas atoms are light enough for the plasma to be produced at the *pulse front*; in other words, we must work under the condition $eE(t_{z_{\max}})/(m\omega_0) \ll c$, where $t_{z_{\max}}$ is the time at which the ions with the charge number $z_{\max} - 1$ are ionized at the highest rate and $z_{\max}$ is the maximum charge of the ions that can be produced during the ionization of a given gas by a given laser pulse. Of course, this restriction does not refer to nonrelativistic pulses. Under the condition $eE(t_{z_{\max}})/(m\omega_0) \ll c$, we can also assume that $|\mathbf{p}_*/(mc)| \ll 1$.

3. ENERGY AND MOMENTUM OF THE IONIZATION-PRODUCED ELECTRONS

The ensemble-averaged momentum $\mathbf{P}(t)$ and energy $Q(t)$ transferred from the laser field to the electrons that originate by the time t in the vicinity of the point x under consideration, both divided by the number of these electrons, are equal to

$$\mathbf{P}(t) = N^{-1}(t)\int_{-\infty}^{t}\int \mathbf{P}(t, t^*, \mathbf{p}_*)\Gamma(t^*, \mathbf{p}_*)d^3\mathbf{p}_* dt^*,$$
$$Q(t) = N^{-1}(t)\int_{-\infty}^{t}\int Q(t, t^*, \mathbf{p}_*)\Gamma(t^*, \mathbf{p}_*)d^3\mathbf{p}_* dt^*, \tag{5}$$

Here, $\mathbf{P}(t, t^*, \mathbf{p}_*)$ and $Q(t, t^*, \mathbf{p}_*)$ are the instantaneous (at the time t) momentum and energy of an electron that originates with the momentum $\mathbf{p}_*$ at the time t^* and $\Gamma(t, \mathbf{p}_*) = \sum_{k=1}^{z_n}\Gamma_k(t, \mathbf{p}_*)n_{k-1}(t)$ where Γ_k and n_{k-1} are determined by formulas (2)–(4).

Assuming that the laser-field envelope changes insignificantly over the laser field period[4] and applying the approach described in [21] (see also [22]), we can write the instantaneous (at the time t) momentum and energy of an electron in the field of a plane, elliptically polarized laser wave (the wave parameters are assumed to depend on the variables x and t only through the combination $x - ct$) in terms of the longitudinal (along the x-axis) displacement $\delta(t, t^*)$ of the electron from the point x at which it is born at the time t^*. Note that, by definition, we have $\delta(t^*, t^*) = 0$.

The kinetic energy $Q(t, t^*, \mathbf{p}_*)$ of an electron originating with the momentum $\mathbf{p}_*$ at the time t^* is related to the projection of its momentum onto the propagation direction of the pulse (the x-axis) by

$$Q(t, t^*, \mathbf{p}_*) = cP_x(t, t^*, \mathbf{p}_*) + mc^2(\kappa_* - 1), \tag{6}$$

where $\kappa_* = \gamma_* - p_{x*}/(mc)$, $\gamma_* = \sqrt{1 + (\mathbf{p}_*/mc)^2}$, and $\mathbf{p}_*^2 = p_{x*}^2 + p_{y*}^2 + p_{z*}^2$.

Now, the momentum of an electron can be written as

$$\mathbf{P}(t, t^*, \mathbf{p}_*) = \mathbf{P}_{\text{van}}(t, t^*, \mathbf{p}_*) + \mathbf{P}_{\text{fin}}(t^*, \mathbf{p}_*),$$

where the components $\mathbf{P}_{\text{van}}$ and $\mathbf{P}_{\text{fin}}$ have the form

$$P_{y_{\text{fin}}} = p_{y*}(\kappa_*/\gamma_*)^2 + 2\kappa_*\sqrt{mQ_p(\phi^*)}\sin(\phi^*),$$
$$P_{y_{\text{van}}} = -2\kappa_*\sqrt{mQ_p(\phi)}\sin(\phi), \tag{7}$$

$$P_{z_{\text{fin}}} = p_{z*}(\kappa_*/\gamma_*)^2 - 2\eta\kappa_*\sqrt{mQ_p(\phi^*)}\cos(\phi^*),$$
$$P_{z_{\text{van}}} = 2\eta\kappa_*\sqrt{mQ_p(\phi)}\cos(\phi), \tag{8}$$

$$P_{x_{\text{fin}}} = 2\frac{\kappa_*}{c}Q_p(\phi^*)[\sin^2(\phi^*) + \eta^2\cos^2(\phi^*)]$$
$$+\frac{1}{2}\left[\frac{\kappa_*^3}{\gamma_*^4}\left(\frac{p_{y*}^2}{mc} + \frac{p_{z*}^2}{mc}\right) + \frac{mc}{\kappa_*} - mc\kappa_*\right]$$
$$+ 2\sqrt{\frac{Q_p(\phi^*)}{mc^2}}\frac{\kappa_*^2}{\gamma_*^2}[p_{y*}\sin(\phi^*) - 2\eta p_{z*}\cos(\phi^*)],$$
$$P_{x_{\text{van}}} = 2\frac{\kappa_*}{c}Q_p(\phi)[\sin^2(\phi) + \eta^2\cos^2(\phi)] \tag{9}$$
$$- 2\frac{\kappa_*}{c}\sqrt{Q_p(\phi)Q_p(\phi^*)}[\sin(\phi)\sin(\phi^*)$$
$$+ \eta^2\cos(\phi)\cos(\phi^*)],$$

Here, $\phi^* = \omega_0 t^*$ is the field phase at which an electron originates in the vicinity of the point x under consideration, $\phi = \omega_0(t - \delta(t, t^*)/c)$ is the field phase at the point at which the electron occurs at the time t, and $Q_p(\phi) = m(eE(\phi)/2m\omega_0)^2$ is the averaged oscillatory energy of an electron at the time t. The longitudinal displacement $\delta(t, t^*)$ of an electron from the point at which it is born at the time t^* to the point at which it occurs at the time t satisfies the transcendental algebraic equation presented in Appendix A.

In formulas (7)–(9), the strongly oscillating momentum component $\mathbf{P}_{\text{van}}$ depends on ϕ, while the weakly oscillating component $\mathbf{P}_{\text{fin}}$ depends only on ϕ^* rather than ϕ. As $t \longrightarrow \infty$, we have $\mathbf{P}_{\text{van}} \longrightarrow 0$, because $\phi = t - \delta(t, t^*)/c \longrightarrow \infty$ by virtue of both $d\delta(t, t^*)/dt < c$ and $Q(\phi \longrightarrow \infty) \longrightarrow 0$. Therefore, the component $\mathbf{P}_{\text{van}}$ makes no contribution to the REM. Consequently, the REM is determined by the component $\mathbf{P}_{\text{fin}}$, which is nonzero in the limit $t \longrightarrow \infty$.

Using formulas (5) and taking into account expressions (7)–(9), (3), (4), and (6), we can see that the ensemble-averaged electron momentum and energy satisfy the relationships

$$\mathbf{P}(t) = \mathbf{P}_{\text{van}}(t) + \mathbf{P}_{\text{fin}}(t), \quad Q(t) = Q_{\text{van}}(t) + Q_{\text{fin}}(t).$$

The strongly oscillating components $\mathbf{P}_{\text{van}}(t)$ and $Q_{\text{van}}(t) = cP_{x_{\text{van}}}(t)$ vanish as $t \longrightarrow \infty$, while the weakly oscillating components $\mathbf{P}_{\text{fin}}(t)$ and $Q_{\text{fin}}(t)$ (which oscillate only slightly about their values averaged over the laser field period) remain nonzero after the passage of the pulse. The weakly oscillating components determine the ensemble-averaged momentum and energy that the laser field transfers irreversibly in the nonadiabatic interaction to the electrons produced in the vicin-

[4] Goreslavsky *et al.* [7] showed that this assumption is valid even for ultrashort laser pulses with a duration of about one laser field period.

ity of the point x by the time t during gas ionization. Hence, after the passage of the pulse (at $t \longrightarrow \infty$ for a Gaussian pulse), the components $\mathbf{P}_{\text{fin}}$ and Q_{fin} are just the REM and REE.

Substituting expressions (7)–(9), (3), and (4) into formula (5), we find that the momentum $\mathbf{p}_*$ does not contribute to the projections of the REM $\mathbf{P}_{\text{fin}}$ onto the y- and z-axes and that the contribution of $\mathbf{p}_*$ to the x-component of $\mathbf{P}_{\text{fin}}$ is determined by the small parameter $|\mathbf{p}_*/(mc)|^2 \ll 1$ and can always be neglected. For this reason, the projections of $\mathbf{P}_{\text{fin}}$ onto the coordinate axes,

$$P_{y_{\text{fin}}}(t) = 2\sqrt{m}N(t)^{-1} \times \int_{-\infty}^{t} \sum_{k=1}^{z_n} Q_p^{1/2}(t^*)\sin(\omega_0 t^*) n_{k-1}(t^*) W_k(t^*) dt^*, \tag{10}$$

$$P_{z_{\text{fin}}}(t) = -\eta 2\sqrt{m}N(t)^{-1} \times \int_{-\infty}^{t} \sum_{k=1}^{z_n} Q_p^{1/2}(t^*)\cos(\omega_0 t^*) n_{k-1}(t^*) W_k(t^*) dt^*, \tag{11}$$

$$P_{x_{\text{fin}}}(t) = 2[cN(t)]^{-1} \int_{-\infty}^{t} \sum_{k=1}^{z_n} Q_p(t^*)[\sin^2(\omega_0 t^*) + \eta^2 \cos^2(\omega_0 t^*)] n_{k-1}(t^*) W_k(t^*) dt^*. \tag{12}$$

have the same form as for $\mathbf{p}_* = 0$.

Formula (5) with expressions (6), (3), and (4) yields the following relationship for $Q_{\text{fin}}(t)$:

$$Q_{\text{fin}}(t) = cP_{x_{\text{fin}}}(t) + Q_*(t), \tag{13}$$

where

$$Q_*(t) = \frac{J_H}{N(t)} \int_{-\infty}^{t} \sum_{k=1}^{z_n} \sqrt{\frac{J_H}{J_k}} \frac{E(t^*)}{E_{\text{a.u.}}} [\eta^2 \sin^2(\omega_0 t^*) + \cos^2(\omega_0 t^*)]^{1/2} n_{k-1}(t^*) W_k(t^*) dt^*. \tag{14}$$

In formula (13), the term Q_* accounts for the initial velocity distribution of the ionization-produced electrons; setting $\Gamma_k(\mathbf{p}_*, t) \sim \delta(\mathbf{p}_*)$ in expression (4) gives $Q_* = 0$.

In order to investigate how the quantities $\mathbf{P}_{\text{fin}}$ and Q_{fin} depend on the parameters of the gas and the laser pulse, it is convenient to eliminate oscillating terms in the integrals in expressions (10)–(14) with the ionization rate and electron density determined by formulas (3) and (2), respectively. Below, the mean electron energy and mean longitudinal (along the x-axis) electron momentum as well as the transverse REM will be analyzed separately.

3.1. Averaged Equations for $P_{x_{\text{fin}}}$ and Q_{fin}

We start by investigating the mean electron energy and mean longitudinal electron momentum. To do this, we perform the time integration in formulas (12) and (14) over subintervals $[t'; t' + \pi/(2\omega_0)]$, each is as long as one-quarter of the laser field period. For a laser pulse with arbitrary polarization, the integrals over the subintervals are expressed in terms of Bessel functions. In this case, the quantities under consideration cannot be represented simply as power functions of the electric field strength (see [13]). In order to avoid difficulties (which are not, however, of fundamental importance), we consider two opposite limiting cases in which the integrals over the subintervals $[t'; t' + \pi/(2\omega_0)]$ can be expanded in asymptotic or power series, because, on a time scale of about ω_0^{-1}, the field amplitude and electron density change only slightly, and, for a sufficiently high ionization rate, the parameter $\alpha_k \sim 10^{-1}$ is small.

(i) If the polarization of a laser pulse is far from being circular, $3\alpha_k/(1 - \eta^2) \ll 1$, formulas (12) and (14) reduce to

$$\bar{P}_{x_{\text{fin}}}(t) = \frac{4N^{-1}(t)}{(\hbar\omega_0)^2 c} \sum_{k=1}^{z_n} J_k^3 \int_{-\infty}^{t} \bar{W}_k(t')\bar{n}_{k-1}(t')\alpha_k^2(t') \times \left[\eta^2 + \frac{3}{2}\alpha_k(t')R_k(t')\right] dt', \tag{15}$$

$$\bar{Q}_*(t) = N^{-1}(t) \sum_{k=1}^{z_n} J_k \int_{-\infty}^{t} \bar{W}_k(t')\bar{n}_{k-1}(t')\alpha_k(t') \times \left[1 - \frac{3}{2}\alpha_k(t') + O\{\alpha_k^2(t')\}\right] dt', \tag{16}$$

where the coefficient R_k incorporates the first three terms of the corresponding asymptotic series in α_k (see Appendix B). With good accuracy, we can assume for estimates that this coefficient is equal to 0.8, because, in a wide range of the pulse parameters, it lies between 0.7 and 1, provided that the relative densities n_{k-1} of the ions in different ionization states are nonzero. The ionization rate $\bar{W}_k(t')$ averaged over the laser field period [see expression (3)] has the form

$$\bar{W}_k(t') = W_k(E(t')) \sqrt{\frac{3\alpha_k(t')}{\pi(1-\eta^2)}} \times \left[1 + \frac{3}{2}\alpha_k(t')\left(n_* - \frac{13}{8} + \frac{1/2}{1-\eta^2}\right) + O\{\alpha_k^2(t')\}\right], \tag{17}$$

where the density of the ions in the kth ionization state, $\bar{n}_k$, averaged over the laser field period, is calculated from formulas (2), in which W_k and $W_k(E(t'))$ are determined by expressions (17) and (3), respectively. We emphasize that, in contrast to formula (3), expression (17) should be taken with the field amplitude $E(t')$ rather than with the instantaneous value $|\tilde{\mathbf{E}}(t')|$ of the rapidly oscillating field. Note also that, for linear polarization, the averaged ADK formula taken with the instantaneous laser field (as is the case in [1, 2]) may lead to an REE overestimated by a factor of approximately 1.5 [20].

For $\eta = 0$, the first term in formula (17) gives the ADK formula for the rate of tunneling ionization by a linearly polarized laser field [14].[5] The remaining terms in the asymptotic series are negligible for $1 - \eta^2 \gg 3\alpha_k$. Note that, at a fixed peak intensity I_0, in accordance with formulas (17), (3), and (1), the main dependence of $\overline{W}$ on η in the limit $1 - \eta^2 \gg 3\alpha_k$ under consideration is determined by the factor $E^{3/2 - 2n_*}(1 - \eta^2)^{-1/2} \sim (1 + \eta^2)^{n_* - 3/4}(1 - \eta^2)^{-1/2}$.

(ii) For a laser pulse with nearly circular polarization, $3\alpha_k/(1 - \eta^2) \gg 1$, formulas (12) and (14) reduce to

$$\bar{P}_{x_{\text{fin}}}(t) = \frac{4N^{-1}(t)}{(\hbar\omega_0)^2 c}\sum_{k=1}^{z_n} J_k^3 \int_{-\infty}^{t} \overline{W}_k(t')\bar{n}_{k-1}(t')\alpha_k^2(t') \times\left[\eta^2 + \frac{(1-\eta^2)\breve{R}_k(t')}{2}\right]dt', \quad (18)$$

$$\bar{Q}_*(t) = N^{-1}(t)\sum_{k=1}^{z_n} J_k \int_{-\infty}^{t} \overline{W}_k(t')\bar{n}_{k-1}(t')\alpha_k(t') \times[1 - (1-\eta^2)/2 + O\{(1-\eta^2)^2\}]dt'. \quad (19)$$

Here, the averaged ionization rates $\overline{W}_k$ have the form

$$\overline{W}_k(t') = W_k(E(t'))\left\{1 - \frac{1-\eta^2}{2}\left[\frac{1}{3\alpha_k(t')} - n_* + \frac{1}{2}\right] + \frac{3}{8}(1-\eta^2)^2\left[\frac{1}{18\alpha_k^2(t')} - \frac{1}{3\alpha_k(t')}\left(n_* + \frac{1}{4}\right) + \frac{n_*^2}{2} - \frac{1}{8}\right] + O\{(1-\eta^2)^3\}\right\}, \quad (20)$$

the densities $\bar{n}_{k-1}$ are calculated from formulas (2) and (20), and $W(t')$ is determined by expression (3) with the field amplitude $E(t')$ in place of the instantaneous field $|\tilde{\mathbf{E}}(t)|$. The coefficient $\breve{R}_k$, which accounts for the power series in $1 - \eta^2$, is presented in Appendix B.

[5] In [14], the corresponding formula is misprinted: the numerical factor $\left(\frac{3e}{\pi}\right)^1$ should be raised to a power of 3/2 rather than 1/2.

According to formulas (16) and (19), the correction Q_* (caused by the nonzero initial electron momentum $\mathbf{p}_*$) to the REE Q_{fin} depends weakly on the degree of elliptical polarization of laser radiation; i.e., in both of the above cases, the factors in square brackets are close to unity. For a circularly polarized pulse, the ratio $Q_*/(cP_{x_{\text{fin}}}) \approx (\hbar\omega_0/2J_k)\gamma \ll 1$ is negligibly small. For a linearly polarized pulse, we have $Q_*/(cP_{x_{\text{fin}}}) \approx 0.9\gamma^2$, so that the correction Q_* can be large in the regime close to the regime of tunneling ionization ($\gamma \sim 1$).

Figure 1 illustrates the dependence of the REE $Q_{\text{fin}}(t \longrightarrow \infty)$ on the degree η of elliptic polarization of the pulse. The curves were obtained numerically from formulas (3) and (12)–(14) for different gases (hydrogen, helium, and oxygen). The residual energies in hydrogen, helium, and oxygen were normalized, respectively, to their values $Q_{\text{fin}} = 23$, 600, and 1810 eV in the case of a circularly polarized ($\eta = 1$) laser pulse with the parameters $I_0 = 5 \times 10^{18}$ W cm^{-2}, $\lambda_0 = 0.78$ µm, and $\tau_{FWHM} = 100$ fs. The curves symbolized by open circles reflect the residual energies calculated from the averaged formulas (15)–(17) (the lower curve) and formulas (18)–(20) (the upper curve); we can see that

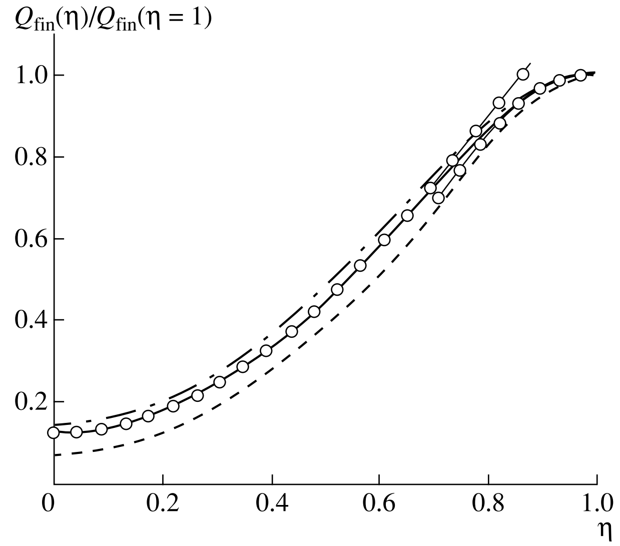

Fig. 1. REE $Q_{\text{fin}}(t \longrightarrow \infty)$, normalized to its maximum (at $\eta = 1$) value, versus the degree η of elliptic polarization of a laser pulse with the parameters $I_0 = 5 \times 10^{18}$ W cm^{-2}, $\lambda_0 = 0.78$ µm, and $\tau_{FWHM} = 100$ fs for hydrogen (dashed-and-dotted curve), oxygen (dashed curve), and helium (solid curve and curves marked by circles). The solid curve illustrates the results obtained from formulas (2), (3), and (12)–(14). The curves marked by circles show the residual energies calculated from formulas (2), (13), and (15)–(17) (the lower curve) and formulas (2), (13), and (18)–(20) (the upper curve). For hydrogen, helium, and oxygen, the maximum residual energies are equal to $Q_{\text{fin}}(\eta = 1) = 23$, 600, and 1810 eV, respectively.

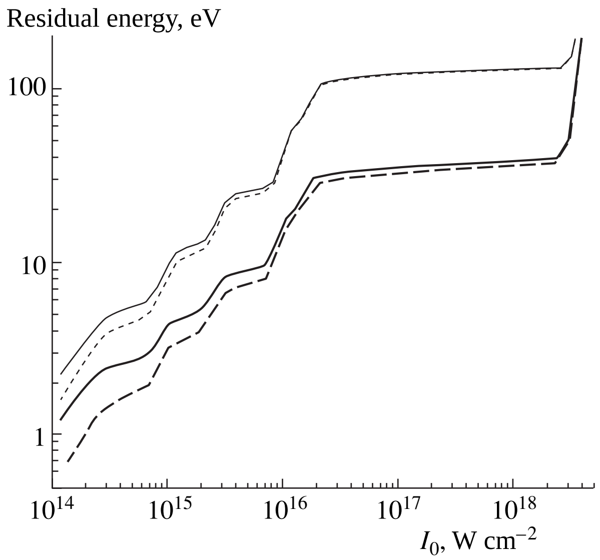

Fig. 2. REE versus the peak intensity I_0 for nitrogen ionized by a laser pulse with the parameters λ_0 = 0.62 μm and τ_{FWHM} = 100 fs. The heavy curves refer to a linearly polarized pulse ($\eta = 0$), and the light curves refer to an elliptically polarized pulse ($\eta = 0.4$). The solid curves are calculated with allowance for the initial velocity distribution of the ionization-produced electrons, and the dashed curves are obtained under the assumption that the electrons originate with a zero initial velocity.

these formulas give quite reliable results for $\eta < 0.8$ and $\eta > 0.8$, respectively.

Since, for the chosen parameter values, the correction Q_* to the main term $cP_{x_{\rm fin}}$ is small, we can assume that the electrons are produced with a zero initial momentum $\mathbf{p}_*$. From Fig. 1, we can also see that the profiles $Q_{\rm fin}(\eta)/Q_{\rm fin}$ ($\eta = 1$) are similar for different gases.

Figure 2 shows the REE as a function of the peak intensity of a laser pulse in nitrogen, calculated from formulas (12)–(14) with and without allowance for the initial velocity distribution of the ionization-produced electrons (in the latter case, the correction Q_* was set at zero). The steps in the dependence of the REE on I_0 correspond to the successive ionization of different electron shells. We can see that the initial velocity distribution of the ionization-produced electrons makes the largest contribution to the REE in the case of low-intensity laser pulses, for which the Keldysh parameter is relatively large. The higher the degree of elliptic polarization, the smaller the contribution of the initial electron velocity distribution to the REE.

Pronounced peaks in the time evolution of the ionization rate $\bar{S} \equiv S_0$ averaged over the laser field period in helium correspond to the successive ionization of different electron shells (Fig. 3). Replacing the peaks by δ-functions, we obtain from formulas (13), (15), (16), and (18) the following estimates:

$$\overline{Q_{\rm fin}}(t) \approx \left[\sum_{k=1}^{Z_{\max}} \theta(t-t_k)\right]^{-1} \sum_{k=1}^{Z_{\max}} \left[\frac{J_k^3}{(\hbar\omega_0)^2}[4\eta^2\alpha_k^2(t_k) + 5\alpha_k^3(t_k)] + J_k\alpha_k(t_k)\right]\theta(t-t_k),$$

$$\frac{3\alpha_k}{1-\eta^2} \ll 1,$$

$$\overline{Q_{\rm fin}}(t) \approx \left[\sum_{k=1}^{Z_{\max}} \theta(t-t_k)\right]^{-1} [1+\eta^2 - 0.7(1-\eta^2)^2] \times \sum_{k=1}^{Z_{\max}} \frac{2J_k^3}{(\hbar\omega_0)^2}\alpha_k^2(t_k)\theta(t-t_k), \tag{21}$$

$$\frac{3\alpha_k}{1-\eta^2} \gg 1,$$

where t_k is the time at which the ionization rate of the ions in the kth ionization state is the highest, $\alpha_k(t_k) \sim 10^{-1}$ is the value of the parameter α_k at the time t_k, $\theta(t - t_k)$ is the Heaviside step function, and $Z_{\max} \le z_n$ is the number of completely ionized electron shells. The coefficients R_k and $\breve{R}_k$ are set to be equal to 0.83 and $1 - 0.7(1 - \eta^2)$, respectively (see Appendix B). As was shown above, the correction $J_k\alpha_k$, which comes from the initial momentum distribution of the ionization-produced electrons, should be taken into account only for laser pulses with a nearly linear polarization.

Estimates (21) imply that, for laser pulses whose polarization is far from being circular ($\eta < 0.8$), the mean energy of the electrons produced from ionization of the kth shell depends on the field amplitude as $Q_{{\rm fin}_k} \propto \eta^2E^2(t_k) + (J_H/J_k)^{3/2}E_{\rm a}^{-1}E^3(t_k)$; for pulses with nearly circular polarization ($\eta > 0.8$), this dependence is $Q_{{\rm fin}_k} \propto E^2(t_k)$. Also, the above formulas show that the mean electron energy is proportional to the squared laser wavelength.

As the peak intensity I_0 of the pulse increases, the point t_k, corresponding to the time at which the ionization rate of the ions in the kth ionization state is the highest, is displaced toward the pulse front along the temporal profile of the pulse. As a result, for peak intensities I_0 that exceed the ionization threshold I_{th} by a factor of two to three, the REE depends only weakly on I_0. (According to [1], the threshold intensity is the pulse intensity at which the potential barrier for an electron in a laser field becomes lower than the ionization potential; for an ion in the $(k-1)$th ionization state, we have

$I_{th} \approx 1.4 \times 10^{14}(J_k/J_H)^4 k^{-2}$ W cm^{-2}.) For multielectron atoms, the REE changes in a jumplike fashion every time the peak intensity of the pulse increases above the threshold for the ionization of each next low-lying electron shell (Fig. 2).

As the laser pulse duration increases or the pulse front becomes less steep [for example, when pulses with a hyperbolic secant envelope are used in place of Gaussian pulses (1)], the point t_k, corresponding to the time at which the ionization rate of the kth electron shell is the highest, is displaced toward the pulse front along the temporal profile of the pulse. As a result, the REE decreases. However, for longer laser pulses such that the ionization front is longer than ten laser field periods, the REE changes insignificantly as the pulse duration increases. Thus, for a laser pulse with the wavelength $\lambda_0 = 0.78$ μm and the intensity $I_0 = 5 \times 10^{18}$ W cm^{-2}, the REE changes only slightly when $\tau_{FWHM} > 100$ fs.

Our calculations showed that, for laser pulses with a peak intensity above the threshold and a duration longer than a hundred picoseconds, the parameter $\alpha_k(t_k)$ is essentially insensitive to the characteristics of laser radiation. Thus, for helium, this parameter takes on the values $\alpha_1 \sim 0.1$ and $\alpha_2 \sim 0.07$, and, for oxygen, we have $\alpha_1 \sim 0.08$, $\alpha_2 \sim 0.06$, $\alpha_3 \sim \alpha_4 \sim \alpha_5 \sim 0.05$, and $\alpha_6 \sim 0.04$. Having found α_k from Eq. (2) with expression (3) or from relationships (17) and (20), we can use formula (21) to estimate the mean energy Q_{fin_k} of the electrons produced during ionization of the gas atoms up to the kth ionization state. For example, for a helium gas ionized by a linearly polarized laser pulse with the same parameters as in Fig. 1, we arrive at the estimates $Q_{\mathrm{fin}_1} \approx 30$ eV, $Q_{\mathrm{fin}_2} \approx 110$ eV, and $Q_{\mathrm{fin}} = (Q_{\mathrm{fin}_1} + Q_{\mathrm{fin}_2})/2 \approx 70$ eV, which agree satisfactorily with the results calculated from more exact formulas (12)–(14): $Q_{\mathrm{fin}_1} \approx 27$ eV, $Q_{\mathrm{fin}_2} \approx 122$ eV, and $Q_{\mathrm{fin}} \approx 75$ eV.

3.2. Equations for the Transverse REM

Before we proceed with the examination of the transverse REM $\mathbf{P}_{\perp\infty} \equiv \mathbf{e}_y P_{y_{\mathrm{fin}}}(t \longrightarrow \infty) + \mathbf{e}_z P_{z_{\mathrm{fin}}}(t \longrightarrow \infty)$ as a function of the laser and gas parameters, note that the integrals of rapidly oscillating functions in expressions (10) and (11) differ substantially from zero only when the width L_S of the ionization curve $S(t)$ is no longer than several laser field periods. In fact, the ionization-produced electrons move initially in different directions, depending on the laser field phases at which they are ejected from the atomic shells. The total electron momentum can substantially differ from zero only when the number of electrons propagating in one direction is markedly larger than the number of electrons propagating in the opposite direction. This situation is possible only when laser pulses are sufficiently short and/or sufficiently intense to produce the ionization front with a small width L_S. While the absolute value of the transverse REM is a monotonically decreasing function of the width of the ionization front, the direction of the REM is governed by the laser field phases at the times t_k at which the ions in the kth ionization state are ionized with the highest rate: the vector of the residual transverse momentum of the electrons produced from ionization of the kth shell will rotate[6] about the x-

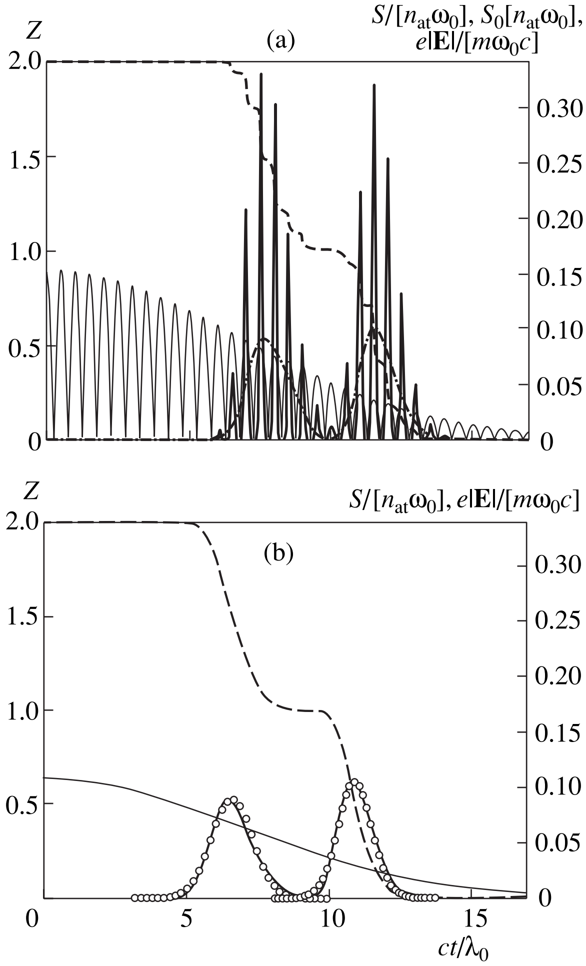

Fig. 3. Dimensionless ionization rate $S(t)/(n_{at}\omega_0)$ (heavy curves), mean ion charge $Z = N/n_{at}$ (dashed and dashed-and-dotted curves), and dimensionless electric field strength $e|\tilde{\mathbf{E}}(t)|/(m\omega_0 c)$ (light curves) for helium ionized by (a) linearly and (b) circularly polarized laser pulses with the parameters $I_0 = 5 \times 10^{16}$ W cm^{-2}, $\lambda_0 = 0.78$ μm, and $\tau_{FWHM} = 30$ fs. In Fig. 3a, the dashed-and-dotted curve is for the zeroth harmonic $S_0 = \bar{S}$ of the ionization rate. In Fig. 3b, the circles illustrate the results from the approximations of the first and second peaks in the ionization curve by Gaussian functions.

[6] For a linearly polarized laser pulse, the transverse REM will reverse direction, because it can be oriented either parallel or antiparallel to the electric field.

axis (along which the pulse propagates) as the field phase at the time t_k will change (as a result of changes in laser pulse parameters).

In order to justify the above considerations and to analytically investigate the dependence of the transverse REM on the laser and gas parameters, we approximate the term S_k, which incorporates the ionization of the kth electron shell into the total ionization rate S in Eqs. (2), by a smooth curve, e.g., such that is described by a Gaussian function $S_k = n_{at}\exp\{-[(t-t_k)/\tau_{S,k}]^2\}/[\sqrt{\pi}\tau_{S,k}]$, where $\tau_{S,k} = L_{S,k}/c$ and $L_{S,k}$ is the characteristic width of the kth ionization front (Fig. 3b). Since the direction of the momentum $\mathbf{P}_{\perp\infty}$ is sensitive to the field phase at the ionization time t_k, we must take into account the phase shift of the oscillating component $\cos(\omega_0 t)$ with respect to the pulse center. For this reason, we specify the electric field of the pulse in the form $\tilde{\mathbf{E}}(t) = E(t)[\mathbf{e}_y\cos(\omega_0 t + \varphi) + \eta\mathbf{e}_z\sin(\omega_0 t + \varphi)]$, in contrast to formulas (1), in which we set $\varphi = 0$. As a result, with allowance for the fact that, on scale lengths $L_{S,k}$, the electric field changes only slightly, we arrive at the expressions

$$P_{y_{\rm fin}}(t \longrightarrow \infty) = 2Z_{\max}^{-1}\sum_{k=1}^{Z_{\max}}\sqrt{mQ_p(t_k)} \times [\sin(\omega_0 t_k + \varphi)e^{-(\omega_0\tau_{S,k}/2)^2}[1-\mu_{k_2}(t_k)] + \sin(3\omega_0 t_k + 3\varphi)e^{-(3\omega_0\tau_{S,k}/2)^2}[\mu_{k_2}(t_k) - \mu_{k_4}(t_k)] + \dots], \tag{22}$$

$$P_{z_{\rm fin}}(t \longrightarrow \infty) = -2\eta Z_{\max}^{-1}\sum_{k=1}^{Z_{\max}}\sqrt{mQ_p(t_k)} \times [\cos(\omega_0 t_k + \varphi)e^{-(\omega_0\tau_{S,k}/2)^2}[1+\mu_{k_2}(t_k)] + \cos(3\omega_0 t_k + 3\varphi)e^{-(3\omega_0\tau_{S,k}/2)^2}[\mu_{k_2}(t_k) + \mu_{k_4}(t_k)] + \dots], \tag{23}$$

where $Z_{\max} \le z_n$ is the number of completely ionized electron shells. The coefficients $\mu_{k_2} \equiv W_{k_2}/(2W_{k_0})$, $\mu_{k_4} \equiv W_{k_4}/(2W_{k_0})$, etc., incorporate high-frequency harmonics in the spectrum of the ionization source for a laser pulse with a noncircular polarization (Fig. 3a). Here, $W_{k_{2n}} = 2\overline{W_k\cos(2\omega_0 t)}$ ($n = 1, 2\dots$) denotes the $[2n]$th high-frequency harmonic of the ionization rate W_k, $W_{k_0} = \overline{W}_k$ is the ionization rate averaged over the laser field period [which is calculated from formula (17) when the pulse polarization is far from being circular and from formula (20) when the pulse polarization is nearly circular], and the superior bar stands for averaging over the laser field period. The first of these coefficients takes on the following values: $\mu_{k_2} = 1 - 3\alpha_k R_k/(1-\eta^2)$ when the pulse polarization is far from being circular, $\mu_{k_2} = 1 - \breve{R}_k \approx 0.7(1-\eta^2) \ll 1$ when the pulse polarization is nearly circular, $\mu_{k_2} \approx 0.75$ for a linearly polarized pulse, and $\mu_{k_2} = 0$ for a circularly polarized pulse. The coefficients $\mu_{k_{2n}}$ with $n > 1$ can be expressed in terms of the analogous power series in α_k or $1-\eta^2$; however, we do not require the exact values of these coefficients, because the first term on the right-hand side of expressions (22) and (23) is much larger than the remaining terms (the exponential functions contain the factor $2n + 1$ and thus rapidly decrease with increasing n).

Formulas (22) and (23) imply that, for singly charged ions (including hydrogen ions) produced by a laser pulse with almost circular polarization ($\mu_{1_2} \ll 1$), the absolute value of the transverse REM,

$$|\mathbf{P}_{\perp\infty}| = \sqrt{P_{y_{\rm fin}}^2(t \longrightarrow \infty) + P_{z_{\rm fin}}^2(t \longrightarrow \infty)} \approx 2\sqrt{mQ_p(t_k)}e^{-(\omega_0\tau_{S,1}/2)^2} \times\sqrt{\eta^2(1+\mu^2) + (1-\mu^2) + [\eta^2(1+\mu^2) - (1-\mu^2)]\cos(2\omega_0 t_1 + 2\varphi)} \tag{24}$$

depends weakly on the phase $\omega_0 t_1 + \varphi$ and decreases exponentially with increasing the ionization front width $\tau_{S,1}$ and, accordingly, the pulse duration τ_{FWHM}. As the field phase changes, the vector of the transverse REM rotates about the x-axis such that the angle θ between $\mathbf{P}_{\perp\infty}$ and the y-axis changes according to the law

$$\theta \approx \arctan\{\eta^{-1}\cot(\omega_0 t_1 + \varphi)(1 + \mu_{1_2}/2)/(1 - \mu_{1_2}/2)\}, \quad \eta \ne 0.$$

For a linearly polarized pulse, the transverse REM is oriented parallel to $\mathbf{E}$ (to the y-axis) and, in accordance with formula (22), changes from about $\approx[-2\sqrt{mQ_p(t_1)}e^{-(\omega_0\tau_{S,1}/2)^2}$ to about $2\sqrt{mQ_p(t_1)}e^{-(\omega_0\tau_{S,1}/2)^2}$ as the field phase at the time t_1 changes. In this case, the angle θ takes on two values: $-\pi/2$ and $\pi/2$.

Note that, according to formula (24), the maximum (for a given width $\tau_{S,1}$ of the ionization front) absolute value of the transverse REM, $|\mathbf{P}_{\perp\infty}|$, is proportional to $\eta(1+\mu_{1_2})$ for $\eta > (1-\mu_{1_2})(1+\mu_{1_2})$ and to $1-\mu_{1_2}$ for $\eta \le (1-\mu_{1_2})(1+\mu_{1_2})$. Consequently, from the asymptotic expressions for μ_{1_2}, we can see that $|\mathbf{P}_{\perp\infty}|$ increases as η increases from 0 (a linearly polarized pulse) to 1 (a circularly polarized pulse).

The above analytic estimates are illustrated by Fig. 4a, which shows the energy of the ordered transverse electron motion, $(2m)^{-1}|\mathbf{P}_{\perp\infty}|^2$, calculated from formulas (10) and (11) for hydrogen ionized by laser pulses with the parameters $I_0 = 5 \times 10^{17}$ W cm^{-2} and $\lambda_0 = 0.78$ μm and with different polarizations $\eta = 0.1$, 0.5, and 1. In accordance with our analytic estimates, the envelopes of the curves $(2m)^{-1}|\mathbf{P}_{\perp\infty}|^2(\tau_{FWHM})$ are exponentially decreasing functions of τ_{FWHM}. The lower the degree η of elliptic polarization of a laser pulse, the more oscillatory is the dependence $(2m)^{-1}|\mathbf{P}_{\perp\infty}|^2(\tau_{FWHM})$; recall that this effect stems from the fact that, as τ_{FWHM} changes, the point t_1 is displaced along the temporal profile of the pulse. The dependence $2m)^{-1}|\mathbf{P}_{\perp\infty}|^2(\tau_{FWHM})$ for helium is shown by the dashed curve in Fig. 4b. We can see that, in contrast to hydrogen, the curve $(2m)^{-1}|\mathbf{P}_{\perp\infty}|^2(\tau_{FWHM})$ for helium decreases nonmonotonically as τ_{FWHM} increases, because the expression for the energy of the ordered transverse electron motion contains the cross terms of the form $\sin(\omega_0 t_k + \varphi)\sin(\omega_0 t_l + \varphi)$ with $k \ne l$ and $\cos(\omega_0 t_n + \varphi)\cos(\omega_0 t_m + \varphi)$ with $m \ne n$, which stem from the summation of the infinite series in the squares of $P_{y_{\rm fin}}$ and $P_{z_{\rm fin}}$ [see expressions (22) and (23)].

Unlike the transverse REM, the longitudinal REM and, accordingly, the REE, which is related to the longitudinal REM by expression (13), experience less pronounced variations as τ_{FWHM} changes. This can be illustrated, e.g., by the dashed-and-dotted and dotted curves in Fig. 4b, which correspond, respectively, to the mean energy $P^2_{x_{\rm fin}}/(2m)(t \longrightarrow \infty)$ of the ordered longitudinal electron motion and the REE $Q_{\rm fin}(t \longrightarrow \infty)$ for helium. That is why, for ultrashort laser pulses, the transverse REM can be much higher than the longitudinal REM. In contrast, for longer pulses, the longitudinal REM becomes higher than the transverse REM, because the latter approaches zero as the pulse duration increases (Fig. 4b).

Our analytic expressions [formulas (24), (21), and (12)–(14)] and calculated results (Fig. 4) also imply that, for the above laser and gas parameters, the mean residual energy $Q_{\rm dir} \equiv [P^2_{x_{\rm fin}}(t \longrightarrow \infty) + |\mathbf{P}_{\perp\infty}|^2]/(2m)$ of the directed electron motion is much lower than the REE $Q_{\rm fin}(t \longrightarrow \infty)$.

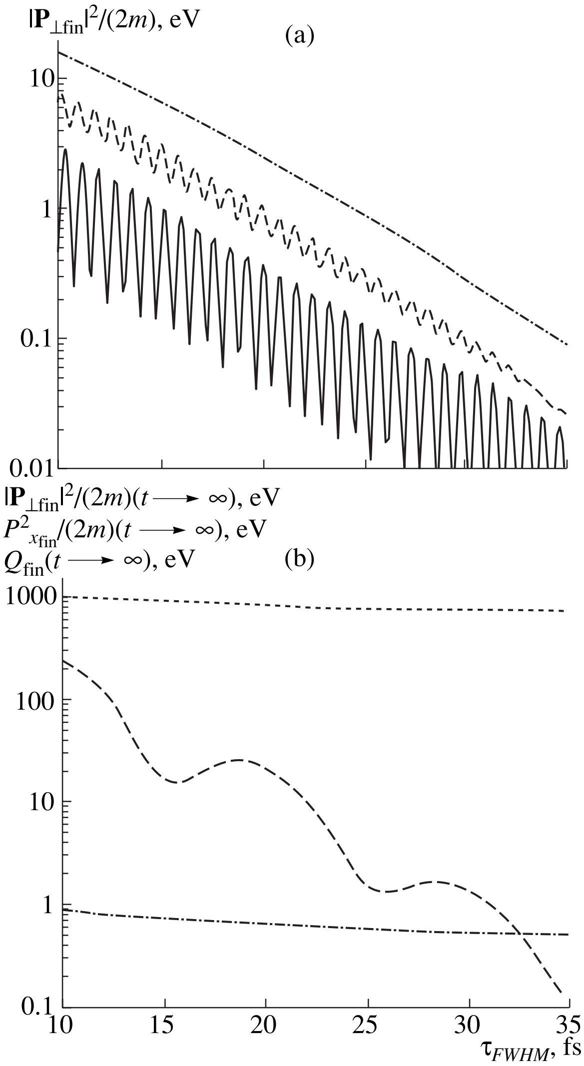

Fig. 4. (a) Dependence of the residual mean energy $|\mathbf{P}_{\perp\rm fin}|^2/(2m)(t \longrightarrow \infty)$ of the ordered transverse electron motion in hydrogen on the duration of laser pulses with $I_0 = 5 \times 10^{17}7$ W cm^{-2} and $\lambda_0 = 0.78$ μm and different polarizations: $\eta = 0.1$ (dashed-and-dotted curve), $\eta = 0.5$ (dashed curve), and $\eta = 1$ (solid curve). (b) Dependence of the residual mean energies $(|\mathbf{P}_{\perp\rm fin}|^2/(2m)(t \longrightarrow \infty))$ (dashed curve), $(P^2_{x_{\rm fin}}/(2m)(t \longrightarrow \infty))$ (dashed-and-dotted curve), and $Q_{\rm fin}(t \longrightarrow \infty)$ (dotted curve) in helium on the duration of a circularly polarized laser pulse with the same parameters.

4. ELECTRON PRESSURE TENSOR IN THE NONRELATIVISTIC COLLISIONLESS LIMIT

Here, we consider the relationship between the REE $Q_{\rm fin}(t \longrightarrow \infty)$ and the electron pressure tensor Π_{ij} or, equivalently, the electron temperature tensor $T_{ij} \equiv n_e^{-1}\Pi_{ij}$. To do this, we use hydrodynamic equations for a low-density gas ionized by the field of a nonrelativistic laser pulse. The desired hydrodynamic equations can be derived from the following collisionless kinetic equation for the electron velocity distribution function $f(\mathbf{r}, \mathbf{v}, t)$:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial r_i} - \frac{e}{m}\frac{\partial}{\partial v_i}\left\{\left(E_i + \frac{e_{ijl} v_j B_l}{c}\right) f\right\} = \sum_{k=1}^{z_n} \Gamma_k(\mathbf{r}, \mathbf{v}, t) n_{k-1}(t), \tag{25}$$

where e_{ijl} is a completely antisymmetric unit tensor, the subscript i stands for the ith vector component, and E_i and B_l are the corresponding components of the electric ($\tilde{\mathbf{E}}$) and magnetic ($\tilde{\mathbf{B}}$) fields of the pulse. We will derive the hydrodynamic equations for the pressure tensor Π_{ij} in the weakly relativistic limit, because, as was shown in the previous section, the REE is insensitive to the relativistic effects if a gas is ionized by a nonrelativistic laser pulse.

Applying the standard method of moments to Eq. (25) yields equations for the hydrodynamic quantities—the electron density $n_e = \int f d^3\mathbf{v}$, electron momentum $\mathbf{P} = n_e^{-1} m \int \mathbf{v} f d^3\mathbf{v}$, and electron pressure tensor $\Pi_{ij} = m\int v_i' v_j' f d^3\mathbf{v}$ (with $v_i' \equiv v_i - V_i$ and $V_i \approx P_i/m$). The corresponding identity transformations put these equations in the form[7]

$$\partial n_e/\partial t + \mathrm{div}(n_e \mathbf{V}) = S, \tag{26}$$

$$\frac{\partial P_i}{\partial t} + V_j \frac{\partial}{\partial x_j} P_i = -eE_i - \frac{P_i S}{n_e} - \frac{e}{c} e_{ijl} V_j B_l - \frac{1}{n_e}\frac{\partial}{\partial r_j}\Pi_{ij}, \tag{27}$$

$$\frac{\partial \Pi_{\alpha\beta}}{\partial t} + \mathrm{div}(\mathbf{V}\Pi_{\alpha\beta} + 2\mathbf{q}_{\alpha\beta}) = mV_\alpha V_\beta S + 2Q'_{\alpha\beta} - \Pi_{\alpha i}\frac{\partial V_\beta}{\partial r_i} - \Pi_{\beta i}\frac{\partial V_\alpha}{\partial r_i} - (e_{\alpha jl}\Pi_{\beta j} + e_{\beta jl}\Pi_{\alpha j})\frac{eB_l}{mc}. \tag{28}$$

Here, $\mathbf{q}_{\alpha\beta} = (m/2)\int v_\alpha' v_\beta' \mathbf{v}' f d^3\mathbf{v}$ is the heat flux vector, the subscripts α and β are fixed, and summation over the repeated Latin indices is used. Using formula (4), we can reduce the quantity $Q'_{\alpha\beta} = m\sum_{k=1}^{z_n} n_{k-1}\int \Gamma_k(\mathbf{v}, \mathbf{r}, t) v_\alpha v_\beta d^3\mathbf{v}$ to

$$Q'_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2(\psi) & -\frac{1}{2}\sin(2\psi) \\ 0 & -\frac{1}{2}\sin(2\psi) & \cos^2(\psi) \end{pmatrix} \times \sum_{k=1}^{z_n} n_{k-1} W_k J_H \sqrt{\frac{J_H}{J_k}}\frac{|\tilde{\mathbf{E}}|}{E_{\mathrm{a.u.}}}, \tag{29}$$

where $\psi(t) = \arctan\{\eta \tan(\omega_0 t)\}$ is the angle between the instantaneous electric field $\tilde{\mathbf{E}}$ and the y-axis.

At the initial time (before the pulse starts to ionize the gas), we have $\Pi_{\alpha\beta}(t = 0) = 0$. For $t > 0$, the tensor $\Pi_{\alpha\beta}$ is determined by the source terms—the first and second terms on the right-hand side of Eq. (28). In order to calculate $\Pi_{\alpha\beta}$ in the first approximation, we can use the smallness of the parameters $|\mathbf{V}/c|$, $|\mathbf{V}/c|^2[S/(n_e\omega_0)]^{-1}$, $e^{-(\omega_0\tau_{S,k})^2}[S/(n_e\omega_0)]^{-1}$ and $|\mathbf{V}/c|[S/(n_e\omega_0)]^{-1}(k_0\sigma_r)^{-1}$ in the ionization region (recall that $\tau_{S,k}$ is the characteristic width of the kth ionization front and σ_r is the characteristic transverse size of the pulse); in Eq. (28), we can also neglect the heat flux vector $\mathbf{q}_{\alpha\beta}$ and the terms containing the combinations of $\Pi_{\alpha\beta}$ and $\mathbf{V}$. When the two subscripts $\{\alpha, \beta\}$ do not run the coordinate pairs $\{\alpha = x, \beta = y\}$, $\{\alpha = y, \beta = x\}$, $\{\alpha = x, \beta = z\}$, or $\{\alpha = z, \beta = x\}$, the last term on the right-hand side of Eq. (28) can also be omitted, because it is proportional to $eB_l/(mc)$. As a result, Eq. (28) becomes

$$\Pi_{\alpha\beta}(t) = \int_{-\infty}^{t} [mV_\alpha(t')V_\beta(t')S(t') + 2Q'_{\alpha\beta}(t')]dt'. \tag{30}$$

For $\{\alpha = x, \beta = y\}$ or $\{\alpha = y, \beta = x\}$, we must supplement the right-hand side of Eq. (30) with the term $-(mc)^{-1}\int_{-\infty}^{t}\Pi_{yy}(t')eE_y(t')dt'$, and, for $\{\alpha = x, \beta = z\}$ or $\{\alpha = z, \beta = x\}$, we must add the term $-(mc)^{-1}\int_{-\infty}^{t}\Pi_{zz}(t')eE_y(t')dt'$. However, expressions (35) (see below) imply that these terms both vanish as $t \longrightarrow \infty$.

In the first approximation, the transverse hydrodynamic velocity components V_y and V_z can be calculated by keeping only the first two terms on the right-hand side of Eq. (27). We integrate this equation by part, neglecting the difference $\partial n_e/\partial t - S = \mathrm{div}(n_e\mathbf{V})$ and assuming that $|\partial \ln|\tilde{\mathbf{E}}|/\partial t|$ is much less than $|\partial \ln n_e/\partial t|$.

[7] Equations (26) and (27) are also valid in the relativistic limit.

As a result, we obtain from (27) the desired transverse velocity components

$$V_y(t) = e(m\omega_0)^{-1}\left[-E(t)\sin(\omega_0 t) + n_e^{-1}(t)\int_{-\infty}^{t} E(t')\sin(\omega_0 t')S(t')dt'\right],$$
$$V_z(t) = \eta e(m\omega_0)^{-1}\left[E(t)\cos(\omega_0 t) - n_e^{-1}(t)\int_{-\infty}^{t} E(t')\cos(\omega_0 t')S(t')dt'\right]. \quad (31)$$

For our purposes, it is sufficient to calculate the longitudinal hydrodynamic velocity V_x to within terms of the second order in the laser field. Under the condition $\chi \equiv S_0/(n_e\omega_0) \ll 1$, we retain the leading-order terms in the Maxwell equations and the equation of motion (27) to arrive at the following expression for V_x:

$$V_x = -(1-\eta^2)(mc)^{-1}Q_p(t) \times [\cos(2\omega_0 t) - \chi\sin(2\omega_0 t)]. \quad (32)$$

Substituting expressions (31) into formula (30), taking into account expressions (10) and (11) for $P_{y_{\text{fin}}}(t)$ and $P_{z_{\text{fin}}}(t)$, and performing identity transformations, we obtain

$$\frac{1}{2}\Pi_{yy}(t) = n_e(t)\left[Q_{y_{\text{fin}}}(t) - \frac{P^2_{y_{\text{fin}}}(t)}{2m}\right],$$
$$\frac{1}{2}\Pi_{zz}(t) = n_e(t)\left[Q_{z_{\text{fin}}}(t) - \frac{P^2_{z_{\text{fin}}}(t)}{2m}\right], \quad (33)$$
$$\frac{1}{2}\Pi_{yz}(t) = \frac{1}{2}\Pi_{zy}(t) = n_e(t)\left[Q_{yz_{\text{fin}}}(t) - \frac{P_{y_{\text{fin}}}(t)P_{z_{\text{fin}}}(t)}{2m}\right],$$

Here,

$$Q_{y_{\text{fin}}}(t) = 2n_e^{-1}(t)\int_{-\infty}^{t}\sum_{k=1}^{z_n} Q_p(t^*)\sin^2(\omega_0 t^*) \times n_{k-1}(t^*)W_k(t^*)dt^* + Q_{yy*}(t),$$

$$Q_{z_{\text{fin}}}(t) = 2\eta^2 n_e^{-1}(t)\int_{-\infty}^{t}\sum_{k=1}^{z_n} Q_p(t^*)\cos^2(\omega_0 t^*) \times n_{k-1}(t^*)W_k(t^*)dt^* + Q_{zz*}(t), \quad (34)$$

$$Q_{yz_{\text{fin}}}(t) = \eta n_e^{-1}(t)\int_{-\infty}^{t}\sum_{k=1}^{z_n} Q_p(t^*)\sin(2\omega_0 t^*) \times n_{k-1}(t^*)W_k(t^*)dt^* + Q_{yz*}(t),$$

where $Q_{\alpha\beta*} \equiv \int_{-\infty}^{t} Q'_{\alpha\beta}(t^*)dt^*$. The energy Q_{fin} determined by formulas (13) and (14) is equal to $Q_{y_{\text{fin}}} + Q_{z_{\text{fin}}} = Q_{\text{fin}}$, and the REE is $Q_{\text{fin}}(t \longrightarrow \infty)$. The sum of the transverse energies, which are associated with distribution (4) of the ionization-produced electrons over their initial velocities, is equal to one-half of the energy Q_* in expression (14): $Q_{yy*} + Q_{zz*} = Q_*/2$. The remaining half is covered by the longitudinal energy, $Q_{xx*} = Q_*/2$.

The last formulas, together with expressions (33) and (34), determine the relationship between the temperature in the (y, z) plane $T_{\mu\nu} = n_e^{-1}\Pi_{\mu\nu}$ (where $\mu, \nu = y, z$), with the REE and REM.

Analogously, using relationships (32) and (31) and taking into account the above additional terms with $\{\alpha = x, \beta = y\}$ and $\{\alpha = x, \beta = z\}$, we obtain from formula (30) the following expressions:

$$\frac{1}{2}\Pi_{xx}(t) = (1-\eta^2)^2\sum_{k=1}^{z_n}\int_{-\infty}^{t}\frac{2Q_p^2(t^*)}{mc^2}[\cos(2\omega_0 t^*) + \chi(t^*)\sin(2\omega_0 t^*)]^2 n_{k-1}(t^*)W_k(t^*)dt^* + Q_{xx*}(t),$$

$$\Pi_{xy}(t) = (1-\eta^2)(mc^2)^{-1/2}\sum_{k=1}^{z_n}\int_{-\infty}^{z_n} Q_p^{3/2}(t^*)[\sin(3\omega_0 t^*) - \sin(\omega_0 t^*) + 2\chi(t^*)\cos(\omega_0 t^*)]n_{k-1}(t^*)W_k(t^*)dt^* - \Pi_{yy}(t)\sqrt{Q_p(t)/(mc^2)}\sin(\omega_0 t), \quad (35)$$

$$\Pi_{xz}(t) = \eta(1-\eta^2)(mc^2)^{-1/2} \times \sum_{k=1}^{z_n}\int_{-\infty}^{t} Q_p^{3/2}(t^*)[\cos(3\omega_0 t^*) - \cos(\omega_0 t^*) - 2\chi(t^*)\sin(\omega_0 t^*)]n_{k-1}(t^*)W_k(t^*)dt^* + Q_{xz*}(t) + \Pi_{zz}(t)\sqrt{Q_p(t)/(mc^2)}\cos(\omega_0 t).$$

Recall that, since $Q_p(t \longrightarrow \infty) \longrightarrow 0$, the additional terms, which are proportional to Π_{yy} or Π_{zz}, vanish as $t \longrightarrow \infty$.

Now, we consider the electron pressure tensor for moderately short and/or moderately intense laser pulses such that the kth ionization front is no shorter than several laser field periods, so that $\omega_0 \tau_{S_k} \gg 1$. For such pulses, formulas (33) and (35) imply that the off-diagonal elements of the pressure tensor as well as the transverse components of the REM are all exponentially small. For laser pulses with different polarizations, the diagonal elements of the pressure tensor and of the temperature tensor $T_{\alpha\alpha} = n_e^{-1}\Pi_{\alpha\alpha}$ can be deduced from formulas (33)–(35) to within unimportant small terms.

(i) When the pulse polarization is far from being circular, $3\alpha_k/(1-\eta^2) \ll 1$, we obtain

$$T_{yy}(t) = 2Q_{y_{\rm fin}}(t) = 12[n_e(t)(\hbar\omega_0)^2]^{-1}$$

$$\times \sum_{k=1}^{z_n} J_k^3 \int_{-\infty}^{t} \overline{W}_k(t')\bar{n}_{k-1}(t')\alpha_k^3(t')\frac{R_k(t')}{1-\eta^2}dt' + Q_*(t)/2,$$

$$T_{zz}(t) = 2Q_{z_{\rm fin}}(t) = 8\eta^2[n_e(t)(\hbar\omega_0)^2]^{-1}$$

$$\times \sum_{k=1}^{z_n} J_k^3 \int_{-\infty}^{t} \overline{W}_k(t')\bar{n}_{k-1}(t')\alpha_k^2(t') \tag{36}$$

$$\times\left[1 - \frac{3}{2}\alpha_k(t')\frac{R_k(t')}{1-\eta^2}\right]dt' + Q_*(t)/2,$$

$$T_{xx}(t) = 8(1-\eta^2)[n_e(t)(\hbar\omega_0)^4 mc^2]^{-1}$$

$$\times \sum_{k=1}^{z_n} J_k^6 \int_{-\infty}^{t} \overline{W}_k(t')\bar{n}_{k-1}(t')\alpha_k^4(t')X_k(t')dt' + Q_*(t),$$

where $\overline{W}_k(t')$ is determined by formula (17) and the asymptotic series for X_k and R_k are presented in Appendix B.

(ii) For a pulse with nearly circular polarization, $3\alpha_k/(1-\eta^2) \gg 1$, we have

$$T_{yy}(t) = 2Q_{y_{\rm fin}}(t) = 4[n_e(t)(\hbar\omega_0)^2]^{-1}$$

$$\times \sum_{k=1}^{z_n} J_k^3 \int_{-\infty}^{t} \overline{W}_k(t')\bar{n}_{k-1}(t')\alpha_k^2(t')\breve{R}_k(t')dt' + Q_*(t)/2,$$

$$T_{zz}(t) = 2Q_{z_{\rm fin}}(t) = 4\eta^2[n_e(t)(\hbar\omega_0)^2]^{-1}$$

$$\times \sum_{k=1}^{z_n} J_k^3 \int_{-\infty}^{t} \overline{W}_k(t')\bar{n}_{k-1}(t')\alpha_k^2(t')[2-\breve{R}_k(t')]dt' + Q_*(t)/2, \tag{37}$$

$$T_{xx}(t) = 4(1-\eta^2)^2[n_e(t)(\hbar\omega_0)^4 mc^2]^{-1}$$

$$\times \sum_{k=1}^{z_n} J_k^6 \int_{-\infty}^{t} \overline{W}_k(t')\bar{n}_{k-1}(t')\alpha_k^4(t')dt' + Q_*(t),$$

where $\overline{W}_k(t')$ is determined by formula (20) and the asymptotic series for $\breve{R}_k$ is presented in Appendix B.

Using expressions (36) and (37), we can estimate T_{xx} as

$$(T_{xx} - Q_*)/Q_*$$
$$\sim 7\times 10^{-2}(1-\eta^2)^2(J_k/J_H)/[(4\gamma)^4(10\alpha_k)].$$

For ions with low charge numbers and for laser pulses with nearly circular polarization, this ratio is, as a rule, smaller than unity; this indicates that the main contribution to the xx-element of the pressure tensor comes from the distribution of the ionization-produced electrons over their initial velocities. For highly ionized atoms and for laser pulses whose polarization is far from being circular, this ratio can be larger than unity, because, in this case, the xx-element of the pressure tensor is governed mainly by the interaction between the laser field and the electrons as they are ejected from the atoms. Formulas (36) and (37) also allow us to conclude that $T_{xx} \ll T_{yy}$ and $T_{xx} \ll T_{zz}$ (the latter is valid for laser pulses whose polarization is sufficiently far from being linear).

Note also that formulas (15), (18), (36), and (37) give

$$\frac{T_{xx} - Q_*}{\bar{P}_{x_{\rm fin}}^2/m}$$
$$\sim \begin{cases} (1-\eta^2)^2/(\eta^2 + 3\alpha_k/2) \gtrsim 1, & 1-\eta^2 \gg 3\alpha_k, \\ (1-\eta^2)^2/2 \ll 1, & 1-\eta^2 \ll 3\alpha_k. \end{cases}$$

5. CONCLUSION

We have investigated the REE and REM in gases ionized by elliptically polarized, relativistic, short laser pulses.

We have shown that, for laser pulses with polarization that is not too close to linear, the distribution of the ionization-produced electrons over their initial velocities is unimportant for obtaining the REE and REM, which thus can be determined under the assumption that the electrons are produced with a zero initial velocity, as is usually done in calculations (see, e.g., [2]). For $\gamma \ll 1$, we can as usual assume that, during ionization of a gas by a linearly polarized laser pulse, the electrons originate with a zero initial velocity. However, at the boundary of applicability range of the tunneling-ionization model ($\gamma \sim 1$), the initial velocity distribution of the ionization-produced electrons may become important for calculating the REE but again has an insignificant influence on the REM.

Analytic formulas (15)–(24) and (33)–(37) make it possible to study how the main parameters of the gas and the laser pulse affect the REE, the REM, and the electron temperature. We have shown that the transverse REM is essentially nonzero only for very short laser pulses (no longer than one or two tens of laser field periods) and decreases exponentially as the pulse duration increases. The same conclusion is valid for the off-diagonal elements of the electron pressure tensor. For longer laser pulses, only the diagonal elements of the pressure tensor are significantly different from zero.

The diagonal elements of the pressure tensor satisfy the inequalities $\Pi_{xx} \ll \Pi_{yy}$ and $\Pi_{xx} \ll \Pi_{zz}$, and the ratio of Π_{yy} to Π_{zz} is determined by the degree of elliptic polarization η (the pulse is assumed to propagate along the x-axis). The REE is expressed in terms of the pressure tensor elements and REM as $Q_{\rm fin} = (2n_e)^{-1}(\Pi_{yy} + \Pi_{zz}) - (2m)^{-1}(P^2_{y_{\rm fin}} + P^2_{y_{\rm fin}})$. If the laser pulse is not too short, the final energy of the directed electron motion, which is proportional to the squared REM, is much lower than the REE.

We have found that the REE is related to the longitudinal REM by the simple expression (13) and is proportional to the third power of the electric field amplitude (at the time of the most intense ionization) for laser pulses with nearly linear polarization and to the second power of the electric field amplitude for pulses with nearly circular polarization. On the other hand, as the peak pulse intensity I_0 changes, the point corresponding to the time at which the ionization rate is the highest is displaced along the temporal profile of the laser pulse. As a result, for the peak intensity I_0 above the ionization threshold for one-electron atoms, the REE depends weakly on I_0, regardless of the pulse shape. For a gas of multielectron atoms, the dependence of the REE on I_0 is jumplike in character, the number of “jumps” being equal to the number of completely ionized electron shells. We have found that the REE is proportional to the squared laser wavelength. We have also shown that the sharper the pulse front, the higher the REE; in particular, the REE is higher for pulses with the same peak intensity I_0 but with a shorter duration.

ACKNOWLEDGMENTS

We are grateful to V. P. Kraĭnov for valuable discussions. This work was supported in part by the Russian Foundation for Basic Research, project no. 98-02-16263. One of the authors (M.E. Veĭsman) also acknowledges the financial support of the International Soros Science Education Program (ISSEP).

APPENDIX A

The displacement δ of an electron from the point at which it is born is described by the equation

$$\begin{aligned}
\delta = (m\omega_0 c)^{-1}\Big[& 2[\omega_0(t-t^*) - k_0\delta]Q_p(t^*) \\
& \times[\sin^2(\omega_0 t^*) + \eta^2\cos^2(\omega_0 t^*)] \\
& + 4Q_p^{1/2}(t^*)Q_p^{1/2}(t-\delta/c)[\sin(\omega_0 t^*)\cos(\omega_0 t - k_0\delta) \\
& - \eta^2\cos(\omega_0 t^*)\sin(\omega_0 t - k_0\delta)] \\
& - (1/2)(1-\eta^2)Q_p(t-\delta/c)\sin(2\omega_0 t - 2k_0\delta) \\
& - (3/2)(1-\eta^2)Q_p(t^*)\sin(2\omega_0 t^*) \\
& + (1+\eta^2)\int_{t^*}^{t-\delta/c} Q_p(\phi)d\phi\Big] + \delta_*(t, t^*, \mathbf{p}_*; \delta),
\end{aligned}$$

$$\begin{aligned}
\delta_*(t, t^*, \mathbf{p}_*; \delta) = {} & [c(t-t^*) - \delta] \\
\times\Bigg[& \frac{2Q_p^{1/2}(t^*)}{\sqrt{m}c}\left(\frac{\mathbf{p}_{y*}}{mc}\frac{\kappa_*}{\gamma_*^2}\sin(\omega_0 t^*) - \eta\frac{\mathbf{p}_{z*}}{mc}\frac{\kappa_*}{\gamma_*^2}\cos(\omega_0 t^*)\right) \\
& + \frac{1}{2}\left(\kappa_*^{-2} - 1 + \frac{\kappa_*^2}{\gamma_*^4}\left[\left(\frac{\mathbf{p}_{y*}}{mc}\right)^2 + \left(\frac{\mathbf{p}_{z*}}{mc}\right)^2\right]\right)\Bigg] \\
& + \frac{2Q_p^{1/2}(t-\delta/c)}{\sqrt{m}\omega_0}\left[\frac{\mathbf{p}_{y*}}{mc}\frac{\kappa_*}{\gamma_*^2}\cos(\omega_0(t-t^*) - k_0\delta)\right. \\
& \left. + \eta\frac{\mathbf{p}_{z*}}{mc}\frac{\kappa_*}{\gamma_*^2}\sin(\omega_0(t-t^*) - k_0\delta)\right] \\
& - \frac{2Q_p^{1/2}(t^*)}{\sqrt{m}\omega_0}\left[\frac{\mathbf{p}_{y*}}{mc}\frac{\kappa_*}{\gamma_*^2}\cos(\omega_0 t^*) + \eta\frac{\mathbf{p}_{z*}}{mc}\frac{\kappa_*}{\gamma_*^2}\sin(\omega_0 t^*)\right].
\end{aligned}$$

In the case at hand, we have $|\mathbf{p}_*/(mc)| \ll 1$. Consequently, for the conditions of tunneling ionization ($\gamma < 1$), we can perform manipulations similar to those in the body of this paper in order to show that $|\delta_*| \ll |\delta - \delta_*|$. Accordingly, in writing Eqs. (7)–(9), we assumed that the displacement δ depends only on t and t^* and is independent of $\mathbf{p}_*$.

APPENDIX B

The coefficient R_k in formulas (15) and (36), the coefficient $\breve{R}_k$ in formulas (18) and (37), and the coefficient X_k in formula (37) for T_{xx} have the form

$$R_k = 1 + \frac{3}{2}\alpha_k\left(2n_* + \frac{1}{1-\eta^2} - \frac{11}{2}\right) + \frac{3\alpha_k^2}{2}\left[\frac{299}{8}\right.$$

$$\left. + 4n_*^2 - 25n_* + \frac{4n_* - 31/2}{1-\eta^2} + \frac{4}{(1-\eta^2)^2}\right] + O\{\alpha_k^3\},$$

$$\breve{R}_k = 1 + \frac{1}{8}\left[2n_* - 1 - \frac{2}{3\alpha_k}\right](1-\eta^2) + O\{(1-\eta^2)^2\},$$

$$X_k = 1 - 6\alpha_k/(1-\eta^2)$$

$$+ (3\alpha_k)^2[(11/2 - 2n_*)/(1-\eta^2) + 2/(1-\eta^2)^2]$$

$$+ O\{\alpha_k^3\}.$$

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Translated by G. V. Shepekina